The Ponzano-Regge asymptotic of the $6j$ symbol: an elementary proof

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Abstract

In this paper we give a direct proof of the Ponzano-Regge asymptotic formula for the Wigner $6j$ symbol starting from Racah's single sum formula. Our method treats halfinteger and integer spins on the same footing. The generalization to Minkowskian tetrahedra is direct. This result should be relevant for the introduction of renormalization scales in spin foam models.

1 Introduction

The connection between the renormalization group, so successful in describing low energy physics, and the theory of loop quantum gravity is still an open question. A promising line of research is to explore in detail the relationship between the renormalization group and the spin foam quantization of gravity [1] (more precisely the group field theory (GFT) dual to spin foams).

GFT [2] can be represented either in terms of group integrals or in terms of tensor models. The situation is highly reminiscent of the one encountered in noncommutative quantum field theory (NCQFT) [3, 4], where the group integral formulation is similar to the direct space representation [5] and the tensorial model is similar to the matrix base representation [6, 7].

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Particularly, the dual GFT of $3D$ quantum gravity resembles a ϕ^4 model. We can describe it in terms of a tensorial field theory, with a vertex weight given by Wigner's $6j$ symbol, and a trivial propagator [2]. This setting, although very encouraging, is not yet adapted to renormalization. The triviality of the propagator makes the definition of scales unclear. The parallel problem in the NCQFT has been solved by the introduction of spectral scales, first in the matrix base [6], and then in the direct space [5]. Consequently, one may argue that the scales in GFT could be more readily accessible in the tensor model formulation. A deeper understanding of this model is then required.

It is well known that the $6j$ symbol obeys the Ponzano-Regge asymptotic formula [8] . Several proofs of this formula exist [9, 10], but they rely either on an algebraic definition or on a group integral definition of the $6j$ symbol.

The goal of this paper is to present an alternative proof of this asymptotic formula, developed entirely in the discrete space of indexes of the tensor model, hence, in a formalism presumably better suited to the introduction of scales and renormalization.

In the next section we give some notations and state our main theorem. The following section consists of its proof. The last section elaborates on the different generalizations of our method.

2 Notations and Main Theorem

There are several ways of expressing the $6j$ symbol. The starting point of our derivation is Racah's single sum formula

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} = \sqrt{\Delta(j_1,j_2,j_3)\Delta(J_1,j_2,J_3)\Delta(J_1,J_2,j_3)\Delta(j_1,J_2,J_3)} \sum_{\max v_i}^{\min p_j} (-1)^t \frac{(t+1)!}{\prod_i (t-v_i)! \prod_j (p_j - t)!} \ , \tag{1}$$

with

$$\begin{aligned} & v_1 = j_1 + j_2 + j_3 \quad v_2 = J_1 + j_2 + J_3 \\ & v_3 = J_1 + J_2 + j_3 \quad v_4 = j_1 + J_2 + J_3 \\ & p_1 = j_2 + J_2 + j_3 + J_3 \quad p_2 = j_1 + J_1 + j_3 + J_3 \quad p_3 = j_2 + J_2 + j_1 + J_1 \\ & \Delta(j_1,j_2,j_3) = \frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \ . \end{aligned} \tag{2}$$

The j's and J's are integers or halfintegers. The sum in eq. (1) is over all integers t such that all the arguments of the factorials are positive. The $\Delta(j_1, j_2, j_3)$ factors are called triangle coefficients.

This weight is associated with an euclidean tetrahedron with edges j_1, j_2, j_3, J_1, J_2, J_3 labeled as in figure 1.

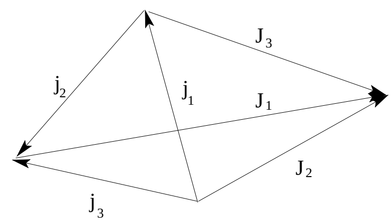

Figure 1: Labeling of the tetrahedron

We can rewrite j_2, J_3, J_1 in terms of j_1, J_2, j_3 like

$$\vec{j_2} = \vec{j_3} - \vec{j_1} \quad \vec{J_3} = \vec{J_2} - \vec{j_1} \quad \vec{J_1} = \vec{J_2} - \vec{j_3}\ . \tag{3}$$

We will prove the following asymptotic formula for the $6j$ symbol

Theorem 1 *Under a rescaling of all its arguments by a large k the $6j$ symbol behaves like*

$$\begin{Bmatrix} kj_1 & kj_2 & kj_3 \\ kJ_1 & kJ_2 & kJ_3 \end{Bmatrix} = \frac{1}{\sqrt{12\pi k^3 V}} \cos\left\{\frac{\pi}{4} + \sum_{i=1}^{3}\left[\left(kj_i + \frac{1}{2}\right)\theta_{j_i} + \left(kJ_i + \frac{1}{2}\right)\theta_{J_i}\right]\right\}. \tag{4}$$

In all sum and products in the sequel the indexes of v run from 1 to 4 while those of p run from 1 to 3. Thus for example $\prod(t - v_i)$ denotes $\prod_{i=1}^{4}(t - v_i)$ whereas $\prod(p_j - t)$ denotes $\prod_{j=1}^{3}(p_j - t)$.

3 Proof of the main theorem

Our proof builds on the techniques developed in [11]. We start by expressing all factorials in eq. (1) by Stirling's formula. Step two consists in approximating the discrete sum over t in eq. (1) by an integral. In step three we give

an asymptotic expression for this integral using a saddle point approximation [1], and obtain theorem 1.

As we are interested only in the dominant behavior we will allways consider only first order approximations, so that troughout this paper = will mean equal up to a multiplicative factor $1 + 1/k$.

3.1 The prefactor

We use Stirling's formula

$$n! = \sqrt{2\pi} e^{(n+\frac{1}{2})\ln(n)-n} \ , \tag{5}$$

to express all the factorials. Thus, a typical triangle coefficient will be

$$\begin{aligned}
&\Delta(kj_1, kj_2, kj_3) = \frac{2\pi}{[k(j_1+j_2+j_3)+1]} \\
&e^{[k(j_1+j_2-j_3)+\frac{1}{2}]\ln[k(j_1+j_2-j_3)]-k(j_1+j_2-j_3)} \\
&e^{[k(j_1-j_2+j_3)+\frac{1}{2}]\ln[k(j_1-j_2+j_3)]-k(j_1-j_2+j_3)} \\
&e^{[k(-j_1+j_2+j_3)+\frac{1}{2}]\ln[k(-j_1+j_2+j_3)]-k(-j_1+j_2+j_3)} \\
&e^{-[k(j_1+j_2+j_3)+\frac{1}{2}]\ln[k(j_1+j_2+j_3)]+k(j_1+j_2+j_3)} \ ,
\end{aligned} \tag{6}$$

were we separated the first term in the denominator. A straightforward computation gives

$$\begin{aligned}
&\Delta(kj_1, kj_2, kj_3) = 2\pi e^{\frac{1}{2}\ln\frac{(j_1+j_2-j_3)(j_1-j_2+j_3)(-j_1+j_2+j_3)}{(j_1+j_2+j_3)^3}} \\
&e^{k[(j_1+j_2-j_3)\ln(j_1+j_2-j_3)+(j_1-j_2+j_3)\ln(j_1-j_2+j_3)+(-j_1+j_2+j_3)\ln(-j_1+j_2+j_3)]} \\
&e^{-k[(j_1+j_2+j_3)\ln(j_1+j_2+j_3)]} \ .
\end{aligned} \tag{7}$$

The prefactor of the sum in eq. (1) is a product of four such triangle coefficients. After some manipulations it can be put into the form

$$(2\pi)^2 e^{H(j,J)+kh(j,J)} \ , \tag{8}$$

[1] The proofs of [9, 10] also use saddle point approximations for some integral representations of the $6j$ symbol.

with

$$
\begin{aligned}
h(j,J) = \quad & j_1 h_{j_1} + j_2 h_{j_2} + j_3 h_{j_3} + J_1 h_{J_1} + J_2 h_{J_2} + J_3 h_{J_3} \\
h_{j_1} = \quad & \frac{1}{2} \ln \left\{ \frac{(j_1 + j_2 - j_3)(j_1 - j_2 + j_3)}{(j_1 + j_2 + j_3)(-j_1 + j_2 + j_3)} \right. \\
& \left. \frac{(j_1 + J_2 - J_3)(j_1 - J_2 + J_3)}{(j_1 + J_2 + J_3)(-j_1 + J_2 + J_3)} \right\} , \qquad (9)
\end{aligned}
$$

and

$$
H(j,J) = \frac{1}{2}(h_{j_1} + h_{j_2} + h_{j_3} + h_{J_1} + h_{J_2} + h_{J_3}) \ . \qquad (10)
$$

3.2 The integral approximation

We now turn our attention to the sum in eq. (1), which we denote by Σ. Denoting $v_1 = j_1 + j_2 + j_3$ etc. (hence without the scale factor k), and separating the $t+1$ term in the numerator, the sum becomes

$$
\Sigma = \frac{1}{(2\pi)^3} \sum_{k \max v_i}^{k \min p_j} e^{g(t)} \ , \qquad (11)
$$

with

$$
\begin{aligned}
& g(t) = \imath\pi t + \ln(t+1) + \frac{1}{2} \ln \frac{t}{\prod(t - kv_i) \prod(kp_j - t)} + t[\ln(t) - 1] \\
& - \sum (t - kv_i)[\ln(t - kv_i) - 1] - \sum (kp_j - t)[\ln(kp_j - t) - 1] \\
& = \imath\pi t + \ln(t+1) + \frac{1}{2} \ln \frac{t}{\prod(t - kv_i) \prod(kp_j - t)} \\
& + t\ln(t) - \sum (t - kv_i) \ln(t - kv_i) - \sum (kp_j - t) \ln(kp_j - t) \ , \qquad (12)
\end{aligned}
$$

where in the last equality we have used $\sum_i v_i = \sum_j p_j$.

We change variables to $t = kx$. The exponent rewrites as

$$
\begin{aligned}
& \frac{1}{2} \ln \frac{x^3}{k^4 \prod(x - v_i) \prod(p_j - x)} + k \Big\{ \imath\pi x + x \ln(x) \\
& - \sum (x - v_i) \ln(x - v_i) - \sum (p_j - x) \ln(p_j - x) \Big\} \ . \qquad (13)
\end{aligned}
$$

Taking out the k in the first logarithm we write the sum as

$$\Sigma = \frac{1}{(2\pi)^3} \sum_{x=\max v_i}^{\min p_j} \frac{1}{k^2} e^{F(x)+kf(x)} \ , \tag{14}$$

where $F(x)$ and $f(x)$ can be read out of eq. (13).

The sum (14) is identified as a Riemann sum. We approximate it by an integral and taking into account that one k^{-1} factor in eq. (14) plays the role of dx we have

$$\Sigma = \frac{1}{(2\pi)^3 k} \int_{\max v_i}^{\min p_j} dx \ e^{F(x)+kf(x)} \ . \tag{15}$$

Note that eq. (7) and (15) can be used to defined a $\{6j\}$ symbol with not only integer and halfinteger entries, but also continuous entries. This continuous version is an analytic continuation of the symbol with (half-)integer entries.

As k is a large parameter the integral (15) can be computes by a saddle point approximation. Taking into account eq. (8) we find the following contribution of a saddle point x_s to the value of the $6j$ symbol

$$\frac{1}{\sqrt{2\pi k^3}} \frac{1}{\sqrt{-f''(x_s)}} e^{H(j,J)+F(x_s)+k[h(j,J)+f(x_s)]} \ . \tag{16}$$

3.3 The Saddle Points

The saddle points equation is

$$f'(x) = \imath\pi + \ln(x) - \sum \ln(x - v_i) + \sum \ln(p_j - x) = 0 \ , \tag{17}$$

that is

$$x(p_1 - x)(p_2 - x)(p_3 - x) = -(x - v_1)(x - v_2)(x - v_3)(x - v_4). \tag{18}$$

The coefficients of x^4 and x^3 compute to zero. The saddle point equation becomes

$$Ax^2 - Bx + C = 0 \ , \tag{19}$$

with

$$
\begin{aligned}
A &= -\sum_{k<l} p_k p_l + \sum_{i<j} v_i v_j = 2(j_1 J_1 + j_2 J_2 + j_3 J_3) \\
B &= -p_1 p_2 p_3 + \sum_{i<j<k} v_i v_j v_k \\
&= 2\big[(j_1 J_1 + j_2 J_2 + j_3 J_3)(j_1 + J_1 + j_2 + J_2 + j_3 + J_3) \\
&+ j_1 j_2 j_3 + J_1 j_2 J_3 + J_1 J_2 j_3 + j_1 J_2 J_3\big] \\
C &= v_1 v_2 v_3 v_4 \ .
\end{aligned} \tag{20}
$$

To solve this equation start by computing its discriminant (denoted by Δ with no arguments)

$$
\begin{aligned}
\frac{4AC - B}{4} = \frac{\Delta}{4} &= j_1^2 J_1^2 (j_2^2 + J_2^2 + j_3^2 + J_3^2 - j_1^2 - J_1^2) \\
&+ j_2^2 J_2^2 (j_1^2 + J_1^2 + j_3^2 + J_3^2 - j_2^2 - J_2^2) \\
&+ j_3^2 J_3^2 (j_2^2 + J_2^2 + j_1^2 + J_1^2 - j_3^2 - J_3^2) \\
&- j_1^2 j_2^2 j_3^2 - J_1^2 j_2^2 J_3^2 - J_1^2 J_2^2 j_3^2 - j_1^2 J_2^2 J_3^2 \ .
\end{aligned} \tag{21}
$$

Substituting all j's and J's in terms of j_1, J_2, j_3 using eq. (3) we find

$$
\frac{\Delta}{4^2} = j_1^2 [\vec{J}_2 \wedge \vec{j}_3]^2 - [\vec{j}_1 \wedge (\vec{J}_2 \wedge \vec{J}_3)]^2 = [\vec{j}_1 \cdot (\vec{J}_2 \wedge \vec{j}_3)]^2 = 6^2 V^2 \ , \tag{22}
$$

where V is the volume of the tetrahedron $j_1, j_2, j_3, J_1, J_2, J_3$!

The two saddle points are then

$$
x_{\pm} = \frac{B \pm \imath\sqrt{\Delta}}{2A} \ . \tag{23}
$$

3.4 The contributions of the saddle points

We use eq. (13) to express f as

$$
f(x) = x \ln\left(\frac{-x\prod(p_j - x)}{\prod(x - v_i)}\right) + \sum v_i \ln(x - v_i) - \sum p_j \ln(p_j - x) \ . \tag{24}
$$

Using eq. (18) we see that, at a saddle point, the first term above is zero. Hence

$$
f(x_{\pm}) = \sum v_i \ln\left(\frac{B}{2A} - v_i \pm \imath\frac{\sqrt{\Delta}}{2A}\right) - \sum p_j \ln\left(p_j - \frac{B}{2A} \mp \imath\frac{\sqrt{\Delta}}{2A}\right) \ . \tag{25}
$$

The real part of f is equal for the two saddle points, hence both are dominant. It is then necessary to take the sum of the two contributions.

We analyze the contribution of t_+. Substituting (2) in (24), $f(t_+)$ writes as a sum over the six numbers j and J

$$f(t_+) = j_1 f_{j_1} + j_2 f_{j_2} + j_3 f_{j_3} + J_1 f_{J_1} + J_2 f_{J_2} + J_3 f_{J_3} \ , \tag{26}$$

where

$$f_{j_1} = \ln \Big[\frac{(x_+ - v_1)(x_+ - v_4)}{(p_2 - x_+)(p_3 - x_+)} \Big] \ . \tag{27}$$

3.5 The second derivative

The second derivative will give a volume factor and an extra piece which we will combine with the $F(x_+)$ term in the exponential. We start by computing the second derivative at the saddle point x_+

$$- f''(x_+) = \sum \frac{1}{x_+ - v_i} + \sum \frac{1}{p_j - x_+} - \frac{1}{x_+} \ . \tag{28}$$

Using the saddle point equation (18) we have

$$\frac{1}{x_+ - v_1} = \frac{-(x_+ - v_2)(x_+ - v_3)(x_+ - v_4)}{x_+ \prod (p_j - x_+)} \ , \tag{29}$$

and substituting the first four factors yields

$$\begin{aligned}
&-f''(x_+) = \frac{1}{x_+ \prod (p_j - x_+)} \\
&\Big[- [(x_+ - v_2)(x_+ - v_3)(x_+ - v_4) + (x_+ - v_1)(x_+ - v_3)(x_+ - v_4) \\
&+(x_+ - v_1)(x_+ - v_2)(x_+ - v_4) + (x_+ - v_1)(x_+ - v_2)(x_+ - v_3)] \\
&+x_+[(p_1 - x_+)(p_2 - x_+) + (p_1 - x_+)(p_3 - x_+) + (p_2 - x_+)(p_3 - x_+)] \\
&-(p_1 - x_+)(p_2 - x_+)(p_3 - x_+)] \Big] \ .
\end{aligned} \tag{30}$$

The numerator of the above fraction computes to

$$\begin{aligned}
&x_+(-2 \sum_{i<j} v_i v_j + 2 \sum_{k<l} p_k p_l) + (\sum_{i<j<k} v_i v_j v_k - p_1 p_2 p_3) \\
&= -2 x_+ A + B = -\imath \sqrt{\Delta} \ .
\end{aligned} \tag{31}$$

Hence

$$-f''(x_+) = \frac{-\imath\sqrt{\Delta}}{x_+ \prod(p_j - x_+)} \ . \tag{32}$$

Substituting eq. (32) into (16) gives

$$\frac{1}{\sqrt{2\pi k^3(-\imath)\sqrt{\Delta}}} e^{H+\frac{1}{2}\ln[x_+\prod(p_j-x_+)]+F(x_+)+k[h+f(x_+)]} \ . \tag{33}$$

Putting together $F(x_+)$ and the contribution given by $f''(x_+)$ we get

$$\begin{aligned}
&\frac{1}{2}\ln[x_+\prod(p_j - x_+)] + F(x_+) = \frac{1}{2}\ln\frac{x_+^4}{\prod(x_+ - v)} \\
&= \frac{1}{2}\ln\frac{\prod(x_+ - v)^3}{\prod(p - x_+)^4} = \frac{1}{2}(f_{j_1} + f_{j_2} + f_{j_3} + f_{J_1} + f_{J_2} + f_{J_3}) \ .
\end{aligned} \tag{34}$$

We conclude that the contribution of the x_+ saddel point is

$$\begin{aligned}
\frac{1}{\sqrt{2\pi k^3(-\imath)\sqrt{\Delta}}} \quad & e^{(kj_1+\frac{1}{2})(h_{j_1}+f_{j_1})+(kj_2+\frac{1}{2})(h_{j_2}+f_{j_2})+(kj_3+\frac{1}{2})(h_{j_3}+f_{j_3})} \\
& e^{(kJ_1+\frac{1}{2})(h_{J_1}+f_{J_1})+(kJ_2+\frac{1}{2})(h_{J_2}+f_{J_2})+(kJ_3+\frac{1}{2})(h_{J_3}+f_{J_3})} \ .
\end{aligned} \tag{35}$$

3.6 Final evaluation

We must compute f_j. We use eq. (27) and compute separately the real and imaginary part. The imaginary part is

$$\begin{aligned}
\Im(f_{j_1}) = \theta_{j_1} \ &= \ \text{Arg}\ (t_+ - v_1) + \text{Arg}\ (t_+ - v_4) \\
&+ \ \text{Arg}\ (p_2 - t_-) + \text{Arg}\ (p_3 - t_-) \ .
\end{aligned} \tag{36}$$

Using eq. (23) we write the four arguments in the above equation as

$$\begin{aligned}
\text{Arg}\ (t_+ - v_1) &= \text{Atan}\Big(\frac{\sqrt{\Delta}}{B - 2Av_1}\Big) \\
\text{Arg}\ (t_+ - v_4) &= \text{Atan}\Big(\frac{\sqrt{\Delta}}{B - 2Av_4}\Big) \\
\text{Arg}\ (p_2 - t_-) &= \text{Atan}\Big(\frac{\sqrt{\Delta}}{2Ap_2 - B}\Big) \\
\text{Arg}\ (p_3 - t_-) &= \text{Atan}\Big(\frac{\sqrt{\Delta}}{2Ap_3 - B}\Big) \ .
\end{aligned} \tag{37}$$

Taking into account that

$$\begin{aligned}&\tan(a_1+a_2+a_3+a_4)\\&=\frac{\sum_i \tan(a_i)-\sum_{i<j<k}\tan(a_i)\tan(a_j)\tan(a_k)}{1-\sum_{i<j}\tan(a_i)\tan(a_j)+\tan(a_1)\tan(a_2)\tan(a_3)\tan(a_4)}\ ,\end{aligned} \quad (38)$$

a straightforward but extremely tedious computation shows that

$$\begin{aligned}&\tan(\theta_{j_1})=\\&\frac{j_1\sqrt{\Delta}}{j_1^2(j_1^2+2J_1^2-j_2^2-J_2^2-j_3^2-J_3^2)+j_2^2J_3^2+j_3^2J_2^2-j_2^2J_2^2-j_3^2J_3^2}\ .\end{aligned} \quad (39)$$

Substituting again j_2, J_3 and J_1 using eq. (3) we find

$$\tan(\theta_{j_1})=\frac{4j_1[\vec{j_1}\cdot(\vec{J_2}\wedge\vec{j_3})]}{4(\vec{j_1}\wedge\vec{j_3})\cdot(\vec{J_2}\wedge\vec{j_1})}=\frac{|(\vec{J_2}\wedge\vec{j_1})\wedge(\vec{j_1}\wedge\vec{j_3})|}{(\vec{J_2}\wedge\vec{j_1})\cdot(\vec{j_1}\wedge\vec{j_3})}\ . \quad (40)$$

As the vectors $\vec{J_2}\wedge\vec{j_1}$ and $\vec{j_1}\wedge\vec{j_3}$ are normal (and outward pointing) to the planes j_1, J_2, J_3 and j_1, j_2, j_3 we identify θ_{j_1} as the (exterior) dihedral angle of the tetrahedron.

We now turn our attention to the real part of f_{j_1}

$$\begin{aligned}\Re(f_{j_1}) &= \ln\Big|\frac{(t_+-v_1)(t_+-v_4)}{(p_2-t_+)(p_3-t_+)}\Big|\\ &= \frac{1}{2}\ln\Big[\frac{[(B-2Av_1)^2+\Delta][(B-2Av_4)^2+\Delta]}{[(2Ap_2-B)^2+\Delta][(2Ap_3-B)^2+\Delta]}\Big]\\ &= \frac{1}{2}\ln\frac{(Av_1^2-Bv_1+C)(Av_4^2-Bv_4+C)}{(Ap_2^2-Bp_2+C)(Ap_3^2-Bp_3+C)}\ .\end{aligned} \quad (41)$$

Again a straightforward but tedious computation shows the real part equals

$$\frac{1}{2}\ln\frac{(j_1+J_2+J_3)(-j_1+J_2+J_3)(j_1+j_2+j_3)(-j_1+j_2+j_3)}{(j_1+J_2-J_3)(j_1-J_2+J_3)(j_1+j_2-j_3)(j_1-j_2+j_3)}\ , \quad (42)$$

and using eq. (9) we conclude that

$$h_{j_1}+\Re(f_{j_1})=0\ . \quad (43)$$

Collecting eq. (35), (40) and (43) yields the following contribution of the x_+ saddle point

$$\frac{1}{\sqrt{48\pi k^3 V}}e^{\imath\frac{\pi}{4}+\imath\sum_{i=1}^3\Big[\Big(kj_i+\frac{1}{2}\Big)\theta_{j_i}+\Big(kJ_i+\frac{1}{2}\Big)\theta_{J_i}\Big]}\ . \quad (44)$$

Summing the constributions of x_+ and x_- proves Theorem 1.

4 Conclusion

Our proof is easily adapted to the Minkowskian tetrahedron. In that case the discriminant changes sign (as the volume becomes imaginary). We find two real saddle points, and only one of the two is dominant. The computations are essentially the same, and it is easy to recover the expected exponential decay.

This method can be genralized to higher $3nj$'s symbols. One needs first to rexpress them as multiple sums and the proceed in a parallel way. For the $9j$ symbol, for instance one should use the three sums formula [12]. The saddle point equations become more involved but the computations should be manageable.

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